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CALCULATION OF VIBRATIONS OF THIN PLATES BY PARTITIONING INTO ELASTICALLY CONNECTED UNDEFORMABLE UNITS

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A mathematical model of a thin plate in the form of a system of elastically connected undeformable rectangular units is proposed and substantiated. With necessary additions, the basic statements of the method can be extended to the case of a thick plate.

Introduction. A mathematical model of a thin plate in the form of a system of elastically connected undeformable rectangular units is particularly attractive owing to its simplicity and the clear physical meaning. This approach has found application in rod calculations [1]. As far as plates are concerned, one can refer to [2]; however, the technique, described in that paper, may be considered only as a first approximation, because a number of fundamental problems, including the problem of accuracy, have not yet been solved. Now we have managed to solve all the major problems. In the present paper, the new results, the approaches, and the general representation of the method are given.

1. Basic Statements. A thin plate of constant thickness is considered; any way of attachment is suitable; point masses can be rigidly attached to the plate; and the orthotropic anisotropy is possible. The plate can have any shape; cuts and notches are admissible. For simplicity, we consider the case where the sections of the contour and the cuts are the segments of straight lines parallel to any of two mutually perpendicular directions.

The plate is divided into N identical rectangular units with the sides dx and dy, connected to each other and to the base by springs of horizontal and vertical action (the plate plane is assumed to be horizontal).

The rigidity of vertical-action springs is denoted by C, and that of horizontal-action springs by S. There are some varieties of springs of both types (Figs. 1 and 2). Figure 1 shows a part of the plate with the unit and generalized-coordinate numbers, and Fig. 2 shows the right edge of one of the units. The springs that connect the adjoining sides of the neighboring units have the rigidities C1 and S1. The rigidities of the springs connecting the corners of the units that are in contact with each other only at their corners are C2 and S2 (here there are two springs of the type S2, one of which acts in the direction of X, and the other in the direction of Y). The rigidities of the springs that connect the edge of the unit to the edges of five "neighbors of its neighbors" and are parallel to these edges are S3, S4, and S5 (the units that are in contact at their corners are also called "neighbors"). Horizontal-action springs are arranged as two layers which are 1/6 of their thickness from the upper and lower planes of the plate. For symmetry, the springs of one layer duplicate those in the other; attachment points of each pair of springs S can be chosen, to a certain extent, arbitrarily. The attachment points of the vertical-action springs are shown in Fig. 2.

The rigidities of the horizontal-action springs (the technique of their determination is described in Sec. 3) are as follows:

$$\mathrm{S1} = 1.28333KD, \quad \mathrm{S2} = -K'D, \quad \mathrm{S3} = 0.10833KD,$$
$$\mathrm{S4} = -0.16666KD, \quad \mathrm{S5} = 0.09375KD,$$

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19 55, 56, 57	20 58, 59, 60	21 61, 62, 63	22 64, 65, 66	17 49, 50, 51
9 25, 26, 27	10 28, 29, 30	11 31, 32, 33	12 34, 35, 36	16 46, 47, 48
5 13, 14, 15	6 16, 17, 18	7 19, 20, 21	8 22, 23, 24	15 43, 44, 45
1 1, 2, 3	2 4, 5, 6	3 7, 8, 9	4 10, 11, 12	14 40, 41, 42
18 52, 53, 54	23 67, 68, 69	24 70, 71, 72	25 73, 74, 75	13 37, 38, 39

Y
X

Fig. 1. Part of the plate (the upper line indicates the numbers of the unit, and the lower line refers to the coordinate numbers).

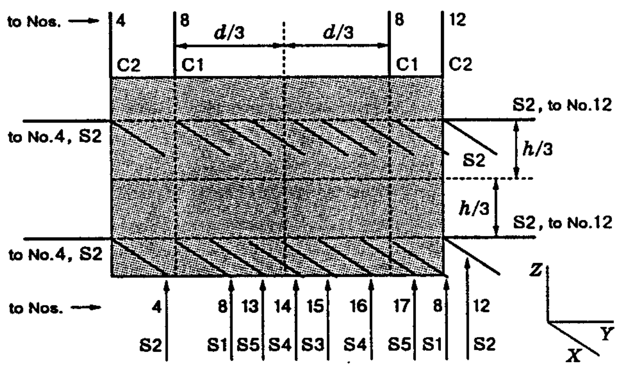

Fig. 2. Springs on the right edge of unit No. 7: figures indicate the numbers of the units to which the second ends of the springs are "attached," h is the thickness of the plate, and d is the length of the edge of the square unit.

where $K = (3 - a^2)/(ah^2)$, $K' = (1.45 - 1.2833a^2)/(ah^2)$, $D = Eh^3/[12(1-\nu^2)]$, $a = b/s$ is the ratio of the length of the unit to its width (by the width, we mean the size of the edge of the unit to which the given spring is attached), E is Young's modulus; ν is the Poisson ratio, and h is the thickness.

The rigidities of the vertical-action springs are as follows:

$$\mathrm{C1} = 2(3 - a^2)(s/b^3)D, \qquad \mathrm{C2} = (2/sb)D.$$

The possibility of negative values is noteworthy, but the sum of the rigidities is of the same order: $\mathrm{C1} + \mathrm{C2} = 6Ds/b^3$ and $\mathrm{S1} + \mathrm{S2} = 2.4D/(ah^2)$.

At the boundary, C1 and C2 are multiplied by the coefficient G2, S1 and S2 by the coefficient G1, and S3, S4, and S5 by the coefficients G3 and G4. The technique of calculating the rigidities and the coefficients G is presented below. The units into which the plate is partitioned are numbered from 1 to N; numbering is continuous. The generalized coordinates are as follows: $q(1)$, $q(4)$, $q(7)$, ... are the displacements of the units' centers; $q(2)$, $q(5)$, $q(8)$, ... are the rotations about the X axis, and $q(3)$, $q(6)$, $q(9)$, ... are the rotations about the Y axis. For example, for the unit with number L, the q coordinates have the numbers $3(L-1)+1$, $3(L-1)+2$, and $3(L-1)+3$.

2. Potential- and Kinetic-Energy Matrices. Frequency Equation. To construct a potential-energy matrix B, it suffices to have the values of the rigidities of the springs and the coordinates of their attachment points. For example, we need to derive an expression for the component $B(1,1)$. We give a virtual displacement $q(1) = 1$ to the system and other displacements are $q(i) = 0$. Then we equate two potential energy expressions for this virtual displacement:

$$\Pi = 1/2\Sigma B(i,k)q(i)q(k), \quad \Pi = 1/2\Sigma C(p,m)g^2(p,m),$$

where $C(p,m)$ is the rigidity of the spring that connects the points p and m; $g(p,m)$ is the difference in the displacements of the points p and m in the direction of spring action. We find $B(1,1)$. Clearly, $B(1+3(i-1), 1+3(i-1)) = B(1,1)$, where $i = 1,2,3,\ldots,N$, except the i which are the numbers of the boundary units. For boundary units, the components $B(i,k)$ are found independently. We give some examples of the values of the components for a square unit with edge d (see Fig. 1):

$$B(19,28) = B(19,34) = B(19,10) = B(19,4) = B(4,52) = \ldots = -\mathrm{C2},$$

$$B(19,19) = B(16,16) = \ldots = 8\cdot\mathrm{C1} + 4\cdot\mathrm{C2},$$

$$B(21,21) = (8/9)h^2\cdot\mathrm{S1} + (13/9)\mathrm{C1}\cdot d^2 + \mathrm{C2}\cdot d^2 + (8/9)h^2\cdot\mathrm{S2} + (4/9)h^2(\mathrm{S3} + 2\cdot\mathrm{S4} + 2\cdot\mathrm{S5})\ldots.$$

It is much easier to construct a kinetic-energy matrix. The matrix is diagonal in the absence of fixed point masses, and the components are the masses and moments of inertia of the units. For example, for unit No. 5 the components are $A(13,13) = M$, $A(14,14) = I_x$, and $A(15,15) = I_y$. If a point load is placed at any point of this unit, the subprogram of the matrix A is supplemented by several rows which indicate its mass and the moments of inertia.

With matrixes A and B, we can solve the frequency equation

$$BQ = \omega^2 AQ, \tag{2.1}$$

where ω is the eigenfrequency and Q is the matrix of oscillation shape.

3. Technique for Determining the Rigidities of Springs. The rigidities of the springs that model the elasticity of the system are presented in Sec. 1. For rigidity calculations, the initial requirement is that the frequency equation (2.1) coincides, within the limit (when the dimensions of the unit tend to zero), with the continuous-plate equation:

$$\Delta\Delta W = \rho\omega^2 hW/D, \tag{3.1}$$

where $W(x,y)$ are the vertical displacements of the plate points and ρ is the density of the plate material. We note that the same equation holds true for derivatives of W with respect to x and y; subsequently, we shall imply precisely this circumstance if formula (3.1) is concerned.

The calculation is performed as follows. We write Eq. (2.1) for a concrete unit. Let it be unit No. 7 (Fig. 1) without point masses:

$$\Sigma B(19,k)q(k) = \omega^2 A(19,19)q(19), \tag{3.2}$$

$$\Sigma B(20,k)q(k) = \omega^2 A(20,20)q(20), \tag{3.3}$$

$$\Sigma B(21,k)q(k) = \omega^2 A(21,21)q(21). \tag{3.4}$$

Let there be a common coordinate system x, y for the plate, and the axes be parallel to the edges of the units. Let the coordinates of the center of unit No. 7 be x and y; the coordinates of the center of unit No. 4 are $(x+dx, y-dy)$, and those of the center of unit No. 13 are $(x+2dx, y-2dy)$, etc. We introduce the following notation: $q(19) = W(x,y)$, $q(20) = U(x,y)$, $q(21) = V(x,y)$, $q(22) = W(x+dx)$, $q(35) = U(x+dx, y+dy)$, etc. Also, we shall present all the generalized coordinates, except $q(19)$, $q(20)$, and $q(21)$, as a power series with respect to powers of dx and dy (with allowance hereinafter that $V = -\partial W/\partial x$ and $U = \partial W/\partial y$); for example, $q(34) = W + W_x\,dx + W_y\,dy + 1/2(W_{xx}\,dx^2 + 2W_{xy}\,dx\,dy + W_{yy}\,dy^2) + 1/6(W_{xxx}\,dx^3 + 3W_{xxy}\,dx^2\,dy + \ldots) + \ldots$. We substitute these representations and the values of $B(i,k)$, expressed through the still unknown C1, ..., S5, into (3.2)–(3.4). Here we assume that C1 and S1 are proportional to the width of the unit and inversely proportional to the cubic length and the length of the unit, respectively (according to the physical meaning of these rigidities, which simulate the cutting force and the tangential stress, respectively); the preliminary assumptions concerning other rigidities are not made. In the resultant equations, we require vanishing the coefficients at the zeroth–third powers dx and dy in (3.2) and at the zeroth–fourth powers in (3.3) and (3.4): in addition, we require that the coefficients at $W_{4,0}$ and $W_{0,4}$ be unity in (3.2) and the coefficient at $W_{2,2}$ be equal to two, and the coefficients at the other fourth derivatives be zero; we also impose similar requirements on the coefficients at the fifth derivatives in Eqs. (3.3) and (3.4). After appropriate calculations, we obtain the values of the rigidities presented in Sec. 1, Eq. (3.2) for a quadratic unit with edge d taking the form [3]

$$\Delta\Delta W + M_6(d/\lambda)^2 + M_8(d/\lambda)^4 + M_{10}(d/\lambda)^6 + \ldots = \rho\omega^2 hW/D \tag{3.5}$$

and Eqs. (3.3) and (3.4) taking the form

$$\Delta\Delta W_{1,0} + L_6(d/\lambda)^2 + L_8(d/\lambda)^4 + L_{10}(d/\lambda)^6 + \ldots = \rho\omega^2 hW_{1,0}/D. \tag{3.6}$$

Here λ is the wavelength with frequency ω. The coefficients L_k and M_k tend to zero as $1/(\lambda^4 k!)$. Thus, the rigidities are chosen so that, with the dimension of the unit tending to zero, Eq. (2.1) takes the form of the continuous-plate equation (3.1). Note that the interaction at the corners of the units and also the "long-range action," i.e., the interaction with the "neighbors of the neighbors," mentioned in Sec. 1, was introduced to obtain the transition (2.1) $\Longrightarrow$ (3.1).

Remark. (1) The unit was assumed to be quadratic in Eqs. (3.3) and (3.4), the transition to a rectangular one was carried out by means of extrapolation with allowance for the values at the "test points."

(2) The character of calculation is completely the same as in Sec. 4.

4. Taking into Account the Boundary Conditions. In the adopted model, the boundary conditions are manifested in the values of the rigidities of the springs at the frontier unit's edge facing the boundary, so that it is required to find the coefficients G by which the values of the rigidities of the boundary unit's springs are multiplied under the given boundary conditions. We solve this problem with the requirement that Eqs. (3.2)–(3.4) take the form of Eq. (3.1) at the plate boundary as the dimensions of the unit tend to zero. We denote the value of the coefficients by G2 for the springs C1 and C2, by G1 for the springs S1 and S2, and by G3 and G4 for the springs S3, S4, and S5. We shall illustrate the calculation. The unit is assumed to be square ($dx = dy = d$). Let unit No. 7 be the boundary, and the boundary is on the right. We write Eqs. (3.2)–(3.4) for this unit and make a preliminary analysis of each equation separately.

(1) Equation (3.2) consists of 14 terms containing coordinates with the numbers 28, 16, 4, 7, 31, 19, 18, 29, 5, 30, 6, 32, 8, and 21; this implies six displacements W of unit No. 7 and its neighbors, four rotations U about the X axis and four rotations V about the Y axis (the number 21 appears because of the different springs on the left and on the right of unit No. 7). The components of the matrix B of Eq. (3.2) have the following values:

$$\begin{gathered} B(19,28) = B(19,4) = -\mathrm{C2}, \quad B(19,7) = B(19,31) = -2\cdot\mathrm{C1}, \quad B(19,16) = -2\cdot\mathrm{C1}, \\ B(19,32) = -B(19,8) = \mathrm{C1}\cdot d, \quad B(19,18) = \mathrm{C1}\cdot d, \quad B(19,29) = -B(19,5) = \mathrm{C2}\cdot d/2, \\ B(19,30) = B(19,6) = \mathrm{C2}\cdot d/2, \quad B(19,19) = 6\cdot\mathrm{C1} + 2\cdot\mathrm{C2} + 2(\mathrm{C1}+\mathrm{C2})\cdot\mathrm{G2}, \\ B(19,21) = (1-\mathrm{G2})(\mathrm{C1}+\mathrm{C2})d. \end{gathered} \tag{4.1}$$

(2) Equation (3.3) contains 18 terms with the coordinates 20, 32, 8, 62, 71, 29, 5, 17, 59, 68, 56, 53, 30, 6, 31, 7, 28, and 4; twelve of them are the rotations about X, four are the displacements W (the numbers 31, 7, 28, and 4), which are the consequence of the rotations of unit No. 7 about the X axis, and two (the numbers 6 and 30) are the rotations about Y, which are the consequence of the rotation of unit No. 7 about X. The components of the matrix B of Eq. (3.3) have the following values:

$$\begin{gathered} B(20,32) = B(20,8) = -4h^2\cdot\mathrm{S1}/9 + \mathrm{C1}\cdot d^2/2, \quad B(20,31) = -B(20,7) = -\mathrm{C1}\cdot d, \\ B(20,62) = B(20,71) = -2\cdot\mathrm{S3}\cdot h^2/9, \quad B(20,29) = \mathrm{C2}\cdot d^2/4 - 2\cdot\mathrm{S2}\cdot h^2/9, \\ B(20,28) = -B(20,4) = -\mathrm{C2}\cdot d/2, \quad B(20,5) = \mathrm{C2}\cdot d^2/4 - 2\cdot\mathrm{S2}\cdot h^2/9, \\ B(20,17) = -2\cdot\mathrm{C1}\cdot d/9, \quad B(20,59) = B(20,68) = -2\cdot\mathrm{S4}\cdot h^2/9, \end{gathered} \tag{4.2}$$

$$B(20,56) = B(20,53) = -2 \cdot \mathrm{S5} \cdot h^2/9, \quad B(20,30) = -B(20,6) = \mathrm{C2} \cdot d^2/4,$$
$$B(20,20) = \mathrm{C1} \cdot d^2 + 2 \cdot \mathrm{C1}(1 + \mathrm{G2})d^2/9 + 8 \cdot \mathrm{S1} \cdot h^2/9 + \mathrm{C2}(1 + \mathrm{G2})d^2/2$$
$$+4(1 + \mathrm{G1})\mathrm{S2} \cdot h^2/9 + 4(\mathrm{S3} + \mathrm{S4} + \mathrm{S5})h^2/9 + 4(\mathrm{S4} + \mathrm{S5})\mathrm{G4} \cdot h^2/9.$$

(3) Equation (3.4) contains 17 terms with the coordinates 19, 33, 9, 18, 21, 30, 6, 15, 27, 3, 57, 54, 29, 5, 16, 4, and 28; eleven of them are the rotations V about the Y axis, four are the displacements W, and two are the rotations U about the X axis. The components of the matrix B of Eq. (3.4) have the following values:

$$B(21,19) = -(\mathrm{G2} - 1)(\mathrm{C1} + \mathrm{C2})d, \quad B(21,33) = B(21,9) = (-2/9)\mathrm{C1} \cdot d^2,$$
$$B(21,18) = -(4/9)h^2 \cdot \mathrm{S1} + (1/2)\mathrm{C1} \cdot d^2, \quad B(21,30) = B(21,6) = (\mathrm{C2}/4)d^2 - 2 \cdot \mathrm{S2} \cdot h^2/9,$$
$$B(21,15) = -2 \cdot \mathrm{S3} \cdot h^2/9, \quad B(21,27) = B(21,3) = -2 \cdot \mathrm{S4} \cdot h^2/9, \tag{4.3}$$
$$B(21,57) = B(21,54) = -2 \cdot \mathrm{S5} \cdot h^2/9, \qquad B(21,29) = -B(21,5) = \mathrm{C2} \cdot d^2/4,$$
$$B(21,16) = -\mathrm{C1} \cdot d, \qquad B(21,4) = B(21,28) = -\mathrm{C2} \cdot d/2,$$
$$B(21,21) = 2h^2(\mathrm{S3} + 2 \cdot \mathrm{S4} + 2 \cdot \mathrm{S5})(1 + \mathrm{G3})/9 + 4 \cdot \mathrm{S1}(1 + \mathrm{G1})h^2/9$$
$$+\mathrm{C1} \cdot d^2(1 + \mathrm{G2})/2 + 4 \cdot \mathrm{C1} \cdot d^2/9 + \mathrm{C2} \cdot d^2(1 + \mathrm{G2})/2 + 4 \cdot \mathrm{S2}(1 + \mathrm{G1})h^2/9.$$

Let $d \to 0$; it is necessary to find the rigidities of the boundary springs by Eqs. (3.2)–(3.4). We turn to Eq. (3.2). Let the coordinates of the center of unit No. 7 be x and y, $q(19) = W(x,y)$, $q(20) = U(x,y) = W_y(x,y)$, and $q(21) = V(x,y) = -W_x(x,y)$. We present the generalized coordinates q of the surrounding units in the form of power series with respect to the powers of dx and dy (see Sec. 3); the components $B(i,k)$ for this case are given in formulas (4.1). After simplifications the equation takes the form

$$12 \cdot \mathrm{G2} \cdot W + 6 \cdot \mathrm{G2} \cdot W_{1,0}d - (W_{3,0} + W_{1,2})d^3 + \left(\frac{W_{4,0}}{2} + W_{2,2} + \frac{5}{6}W_{0,4}\right)d^4 = \rho\omega^2 hd^4 \frac{W}{D}. \tag{4.4}$$

Similarly, substituting the values of (4.2) into Eq. (3.3) and representing the generalized coordinates of the units that are the neighbors of unit No. 7 as a power series, we obtain

$$(22.6666 \cdot \mathrm{G2} - 0.7776 \cdot \mathrm{G4} - 0.888 \cdot \mathrm{G1})W_{0,1} + 10W_{1,1}d - 4W_{0,3}d^2 + (-1.3333W_{3,1} - 4W_{1,3})d^3$$
$$+(0.14444W_{0,5} + 0.5W_{4,1} + W_{2,3})d^4 = \rho\omega^2 hd^4 W_{0,1}/D. \tag{4.5}$$

Similarly, Eq. (3.4) is reduced to the form

$$-72 \cdot \mathrm{G2} \cdot W/d + W_{1,0}(-12.8 \cdot \mathrm{G1} - 36 \cdot \mathrm{G2} + 0.2 \cdot \mathrm{G3}) - 12.4W_{2,0}d + 5.3333W_{1,2}d^2$$
$$+(1.1333W_{4,0} + 4.2222W_{2,2} + W_{0,4})d^3 + (-0.5W_{5,0} - W_{3,2} - 0.0555W_{1,4})d^4 = -\rho\omega^2 hd^4 W_{1,0}/D. \tag{4.6}$$

We recall that in formulas (4.4)–(4.6), W is the displacement of the center of the frontier unit. Naturally, W and the derivatives should be expressed via the value at the plate boundary, which is at a distance of $d/2$ from the center of unit No. 7. Again, we do it by means of power series. The boundary is on the right of unit No. 7 for $x = r$. We introduce the following notation: $W(r,y) = W^0$. For the center of unit No. 7 (Fig. 1), one can write, to within an accuracy comparable with the right-hand side of the equations,

$$W = W^0 - W^0_{1,0}d/2 + W^0_{2,0}d^2/8 - W^0_{3,0}d^3/48 + W^0_{4,0}d^4/(24 \cdot 16) - W^0_{5,0}d^5/(120 \cdot 32),$$
$$W_{1,0} = W^0_{1,0} - W^0_{2,0}d/2 + W^0_{3,0}d^2/8 - W^0_{4,0}d^3/48 + W^0_{5,0}d^4/(24 \cdot 16), \tag{4.7}$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$W_{4,1} = W^0_{4,1}, \quad W_{5,0} = W^0_{5,0}, \quad W_{3,2} = W^0_{3,2}, \dots .$$

We consider the case of a freely supported plate having an exact solution [4, 5]. We assume a "pure" case where the responses of the support have no component in the plate plane. At the boundary, we have $\tau_{xy} = 0$, and hence $W_{1,1}(r,y)=0$. Generally, this equality is not used as the boundary condition; here its consequence $W_{1,2}=0$ appears during the solution [see below the analysis of the results (4.10) and (4.14)]. In what follows, we use the boundary conditions $W(r,y)=W_{2,0}(r,y)=0$ and their consequences $W_{0,1}(r,y)=W_{2,1}(r,y)=W_{2,2}(r,y)=\ldots=0$, because $dW_{i,k}(r,y)/dy=(W_{i,k+1}(x,y))_{x=r}$. We begin with formula (4.6), because it is the most important [this formula concerns rotations about the Y axis, which are different from zero at the boundary of the supported plate; formulas (4.4) and (4.5) refer to the displacements and rotations about X, which are infinitesimal at the boundary]. Taking into account the boundary conditions of a supported plate and their consequences $W^0=W^0_{2,0}=W^0_{0,1}=W^0_{2,1}=\ldots=0$ and substituting the values of (4.7) into formula (4.6), we obtain

$$(-12.8\cdot \mathrm{G1}+0.2\cdot \mathrm{G3})W^0_{1,0}+[(-3\cdot \mathrm{G2}+(-12.8\cdot \mathrm{G1}+0.2\cdot \mathrm{G3})/8+6.2)W^0_{3,0}+5.3333W^0_{1,2}]d^2$$
$$+[9\cdot \mathrm{G2}/16-(-12.8\cdot \mathrm{G1}+0.2\cdot \mathrm{G3})/48-0.4167]W^0_{4,0}d^3+[(-0.06458\cdot \mathrm{G2}$$
$$+(-12.8\cdot \mathrm{G1}+0.2\cdot \mathrm{G3})/384-0.8083)W^0_{5,0}-2.4444W^0_{3,2}-0.5555W^0_{1,4}]d^4=-\rho\omega^2hd^4W^0_{1,0}/D. \tag{4.8}$$

We analyze (4.8). Let $d\to 0$. Assuming that the values of $W^0_{1,0}$, $W^0_{1,2}$, $W^0_{3,0}$, $W^0_{5,0}$, $W^0_{1,4}$, and $W^0_{3,2}$ differ from zero, we have

$$-12.8\cdot \mathrm{G1}+0.2\cdot \mathrm{G3}=0; \tag{4.9}$$

$$(-3\cdot \mathrm{G2}+(-12.8\cdot \mathrm{G1}+0.2\cdot \mathrm{G3})/8+6.2)W^0_{3,0}+5.3333W^0_{1,2}=0; \tag{4.10}$$

$$(-0.06458\cdot \mathrm{G2}+(-12.8\cdot \mathrm{G1}+0.2\cdot \mathrm{G3})/384-0.8083)W^0_{5,0}-2.4444W^0_{3,2}-0.5555W^0_{1,4}=-\rho\omega^2hW^0_{1,0}/D; \tag{4.11}$$

$$W^0_{4,0}=0. \tag{4.12}$$

As is known [4], the value of (4.12) corresponds to the exact solution. It follows that:
(1) G1 = G3 = 0, which is quite natural, because there is no bending moment at the boundary;
(2) From formula (4.10), we have $(-3\cdot \mathrm{G2}+6.2)W^0_{3,0}+5.3333W^0_{1,2}=0$. Taking into account the independent results (4.13) and (4.14), where the expression in brackets should be equal to zero for d^3, we obtain $W^0_{3,0}=W^0_{1,2}=0$, and hence $W^0_{3,2}=0$ and $W^0_{1,4}=0$; these values were used in (4.11);
(3) With allowance for the aforesaid, from formula (4.11) we have $-(0.06458\cdot \mathrm{G2}+0.8083)W^0_{5,0}=-\rho\omega^2hW^0_{1,0}/D$.

To reach the coincidence with (3.1), the coefficient at $W^0_{5,0}$ should equal unity; in this case, G2 = 2.968.

We substitute the values (4.7), the boundary conditions, and the just obtained value of G2 into Eq. (4.5). Since the boundary behaves as a rigid restraint in rotations about the X axis, we have G1 = 2. The equation takes the form

$$(-22.7493+0.3888\cdot \mathrm{G4})W^0_{1,1}d-((1.4479+0.0162\cdot \mathrm{G4})W^0_{3,1}-2W^0_{1,3})d^3+0.1444W^0_{0,5}d^4=\rho\omega^2hd^4W^0_{0,1}/D. \tag{4.13}$$

At the boundary $x=r$, we have $W^0_{0,5}=W^0_{0,1}=0$. It is evident that vanishing the expressions in the first two brackets is required. Vanishing the first brackets gives a value of G4 = 58.5116. With allowance for Eq. (4.10) and the remarks on the derivatives with respect to y at the boundary, vanishing the second brackets produces $W^0_{3,1}=W^0_{1,3}=0$.

Thus, the desired coefficients are found. We should analyze Eq. (4.4). Substituting (4.7) into (4.4) and taking into consideration the boundary conditions, we obtain

$$((\mathrm{G2}-1)W^0_{3,0}-W^0_{1,2})d^3+(1-3\cdot \mathrm{G2}/32)W^0_{4,0}d^4=\rho\omega^2hd^4W^0/D. \tag{4.14}$$

Taking into account the previous results, we see that the equation is satisfied. The analysis is completed. Among the results obtained, $W^0_{3,0}(r,y)=0$ is noteworthy. It means that the transverse force is equal to zero at the boundary. Apparently, it is admissible, because "... being applied to an edge lying on a support, transverse forces exert a certain effect only on the magnitude of the basic responses of the plate and on the distribution of the stresses in it near this edge" [4, p. 305].

TABLE 1

Values of the First Ten Frequencies of a Supported Square Plate for Different Numbers of Partitions into Elastically Connected Units

N	$\bar{\omega}$									
2 × 2	52.2689	50.6050	48.9105	48.5983	46.4398	45.4073	35.0027	30.2733	30.2716	15.7902
3 × 3	121.2279	109.4774	106.5355	98.8394	79.9424	75.8424	67.7897	63.5578	44.9527	20.9561
4 × 4	139.3658	135.8520	127.1553	121.9488	116.7213	105.0009	85.3219	72.3561	47.2248	20.0488
5 × 5	183.0714	179.9596	160.5345	149.5438	139.3917	114.7380	90.4010	73.8870	47.6664	19.9033
6 × 6	208.2786	206.9270	173.8288	158.7100	147.9784	118.4440	92.4402	74.9737	48.0864	19.8417
7 × 7	220.8399	218.8231	179.8524	163.2577	152.9525	120.6529	93.9923	76.0880	48.3702	19.8408
8 × 8	228.7611	225.1277	183.3627	166.0366	156.0852	122.0888	95.0193	76.6779	48.5535	19.8100
9 × 9	234.1175	228.9915	185.8173	167.9686	158.2592	123.1168	95.7284	77.0773	48.6832	19.7865
10 × 10	237.9656	231.6515	187.6468	169.3859	159.8507	123.8830	96.2389	77.3630	48.7794	19.7706
Exact value	256.6097	246.7401	197.3921	177.6529	167.7833	128.3048	98.6960	78.9568	49.3480	19.7392

The values obtained for G1–G4 were introduced into the program. Table 1, which lists theoretical [6] and calculation results for the dimensionless frequencies $\bar{\omega} = \omega a^2(\rho h/D)^{1/2}$ are given for a supported square plate with edge a for various partitions of the plate into unit, shows the accuracy of calculation and the convergence of the results

The same method was employed in the case of a restrained boundary. The approach in which the calculated contour of the plate was at a distance equal to half the length of the unit from the actual contour of restraint is more precise. For this case, G1 = G2 = 1, G3 = 2, and G4 = 2.2874. The accuracy is the same as in the case of a supported plate. A comparison is made with the COSMOS calculation with a 10 × 10 partition in both cases. For the lowest frequencies, the considered approach is more accurate.

The case of a free boundary is the simplest, because there are no "springs" at the boundary. The solution is more correct if the partition boundary is extended beyond the plate at a distance equal to half the length of the unit.

The frequencies and amplitude-frequency characteristics of a rectangular glass textolite plate attached at four and six points were studied experimentally and computed in [7]. Agreement of the results is quite satisfactory.

5. Classification and the Value. Our approach has common features with the finite-difference and finite-element methods (FDM and FEM), but is not a variant of the first or the second. We make an attempt to justify this statement. First of all, we would like to make the following remark, which is of the fundamental character. As is known, in the adopted theory of thin plates, in deriving the differential equation of motion of a plate the stress-tensor components τ_{xz} and τ_{yz} are ignored, and it is quite natural owing to the adopted expression of the strain-tensor components via the displacement W. Ignoring is reached by expressing these components via other components in the two equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \qquad \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = -\rho \frac{\partial^2 W}{\partial t^2 h}$$

and by substituting into the third equation, so that one differential equation is derived (in the FDM, this equation produces a system of algebraic finite-difference equations). Our approach does not permit us to do this. In our approach, the springs that should be presented explicitly, namely, via their rigidities and the attachment points, correspond to the stress-tensor components. But it is natural that the departure from the adopted theory appeared in solving this problem, whereas the FDM and the FEM are fully guided by this theory. This made it necessary to take into account the interaction at the corners of the units and the "long-range action," i.e., at a definite stage our approach deviates from the adopted theory. But later the methods "converge" again: each of three equations in our approach takes the form of the differential equation of a plate (we recall that this is reached in the limit for an infinitely increasing number of units in the process of determination of the rigidities of the springs).

We have already mentioned that there is a formal need to introduce the "long-range actions," because it was impossible to reduce (within the limit) algebraic equations to a differential equation of a continuous plate ignoring this introduction. However, this technique also has a definite physical meaning. Indeed, the "springs" simulate the total elasticity of a given element (say, unit No. 7 in Fig. 1) and its surroundings. It seems completely natural to simulate this elasticity by a method of successive refinements: first, to use four neighbors that are in contact with its edges as the surrounding of this unit (the first approximation) and then to include the units that are in contact with unit No. 7 only at the corners (the second approximation), and, finally, to include the outer ring in Fig. 1 in this surrounding (the third approximation). The inclusion of the following ring embracing the previous ring is possible (the fourth approximation) by introducing appropriate springs; the rigidities of these springs are found from the requirement that the second terms on the left-hand side of Eqs. (3.5) and (3.6) vanish. One can continue unboundedly, expanding the region forming the elastic interaction of the given unit with the surrounding material of the plate.

We continue a comparison with the known methods.

1. Common with the FEM is partition into units. However, in the FEM the units are deformable and should be boundary-shared; in our approach, the units are rigid, and they are connected to each other and to the boundary by springs, and conjugation with the neighboring elements is not needed. Consequently, the mathematical models are developed differently..

2. The similarity to the FDM consists in the use of Taylor series and the fact that the finite differences in the FDM and the increment of the displacements W of the neighboring elements in our approach are as a matter of fact the same. But the similarity is limited by these, purely external features. In essence, the methods are different. Indeed, the FDM originates from Eq. (3.1), and a system of N algebraic equations is obtained for the determination of the displacement of N nodes in partitioning into N nodes. In our approach, system (2.1) of $3N$ linearly independent equations is initial for the determination of $3N$ unknowns, namely, the displacements and rotations of N elements. Here the rigidities of the springs are found from the condition that each of three equations in (2.1) coincides to within the limit with (3.1). It is difficult to present how it is possible to formally reduce one problem to the other, the FDM to our approach [taking into account that $3N$ equations of (2.1) are independent, whereas Eq. (3.1) for W and the corresponding two equations for W_x and W_y can hardly be reduced to $3N$ independent algebraic equations, if at all]. As for the inclusions of W_x and W_y in the number of unknowns in our approach, it is important for a study of the stress state and, probably, increases the computational accuracy. In addition, our approach is able not to use the second ring of the surrounding: if one removes the appropriate springs S3, S4, and S5, one obtains the second approximation which is quite acceptable in accuracy, whereas the FDM cannot work without this ring. And one more example. We consider the boundary conditions in our approach and in the FDM in the case of a free boundary. In the FDM, the boundary conditions not only require the entry to the second ring of the surrounding of the boundary unit, but also are expressed in a rather complicated way, whereas in the proposed approach, the springs at the free boundary of the boundary unit are just removed and their rigidities are equated to zero.

3. The important difference between our approach and the FDM and FEM is the clear physical meaning.

4. The possibilities of the methods also differ. The proposed approach allows one to locate the point mass not only in the center of the unit, but also at any point of the unit by adding three rows in the matrix A. In real problems, the "rigid" boundary conditions of the usual standards should be often "weakened." In the given approach, it is done rather simply, by changing the coefficients G on the necessary section of the boundary.

The approach considered is extended to the case of large sags. For this purpose, it is required to return to the units the "taken away" degrees of freedom, namely, the displacements along X and Y (and rotations in the XY plane for some problems). The springs remain the same, but their declination should be taken into account, and therefore the components of the matrix B depend on the displacements and rotations of the unit. Also, the method can be extended to the case of a thick plate. To do this, it is necessary:

(1) To present a thick plate as a set of thin plates divided into elastically connected undeformable units and to introduce the elastic connection between the units of various plates;

(2) To give three additional degrees of freedom to each unit;

(3) To take into account the additionally introduced linkages (compared to a unitary thin plate) in the second approximation of the proposed approach.

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